

# Quantum scattering of charged solitons in the complex sine-Gordon model

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## Abstract

The scattering of charged solitons in the complex sine-Gordon field theory is investigated. An exact factorizable S-matrix for the theory is proposed when the renormalized coupling constant takes the values  $\lambda_R^2 = 4\pi/k$  for any integer  $k > 1$ : the minimal S-matrix associated with the Lie algebra  $a_{k-1}$ . It is shown that the proposed S-matrix reproduces the leading semiclassical behaviour of all amplitudes in the theory and is the minimal S-matrix which is consistent with the semiclassical spectrum of the model. The results are completely consistent with the description of the complex sine-Gordon theory as the  $SU(2)/U(1)$  coset model at level  $k$  perturbed by its first thermal operator.

# 1 Introduction

The idea that extended particles in quantum field theory can be associated with soliton solutions of the corresponding classical field equations dates back to the early work of Skyrme [1]. The sine-Gordon (SG) field theory is an exactly solvable model which provides an important paradigm for this idea. However, the sine-Gordon kink lacks a key feature which is present in more realistic four-dimensional theories: it has no internal degrees of freedom. All solitons, including the SG kink, have a translational degree of freedom which corresponds to the position of the centre of mass, but the soliton solutions of a field theory which has an unbroken global symmetry will also have internal collective coordinates. When the theory is quantized these coordinates give rise to a tower of massive particles which carry the corresponding conserved charge. One example of this phenomenon occurs in the Skyrme model, where the space of static soliton solutions is parametrized by an  $SU(2)$  collective coordinate. The corresponding particles have angular momentum and isospin quantum numbers and are identified with the baryons of large- $N_c$  QCD [2]. Another example is the 't Hooft-Polyakov monopole in non-abelian gauge theory [3, 4]. The monopole has an internal  $U(1)$  degree of freedom which gives rise to dyons; particles which carry both electric and magnetic charge [5].

In both the examples mentioned above, the spectrum and interactions of these particles can generally only be found at weak coupling and so it would be useful to have a simple model with internal symmetry which was also exactly solvable in the sense of the SG theory. In this paper we will study such a model and propose an exact solution under certain conditions. The complex sine-Gordon (CSG) theory [6, 7] is defined by the following Lagrangian in two spacetime dimensions which has a global  $U(1)$  symmetry:

$$\mathcal{L} = \frac{|\partial_\mu \psi|^2}{1 - \lambda^2 |\psi|^2} - m^2 |\psi|^2, \quad (1)$$

where  $\psi = \psi_1 + i\psi_2$  is a complex scalar field. Like the more familiar SG equation, the classical field theory which follows from (1) has both single soliton solutions and multi-soliton scattering solutions which can be constructed analytically [7, 9]. In the classical theory, the solitons exhibit completely elastic scattering, the only effect of which is a time-delay relative to free motion. By analogy with the SG case one might hope that the corresponding quantum theory would also be completely integrable and would therefore have a factorizable S-matrix. De Vega and Maillet [8] calculated the scattering amplitude for the elementary particles of the theory (which we will call mesons) in ordinary perturbation theory and found that the S-matrix is indeed factorizable at tree level. At the one-loop level however, the factorizability of the theory breaks down, and can only be regained if a local counter-term is added to the bare Lagrangian. It is not known if this procedure can be carried out to all orders to yield an exactly factorizable S-matrix for the mesons. In this paper we will be led to the surprising result that the theory only admits a factorizable S-matrix for particular values of the renormalized coupling; namely  $\lambda_R^2 = 4\pi/k$  where  $k$  is an integer greater than 1. For these values we will propose an exact S-matrix for the CSG theory which is consistent both with the general requirements of completely elastic scattering theory in two dimensions [10] and with the semiclassical limit of the model. The S-matrix in question is

the minimal S-matrix associated with the Lie algebra  $a_{k-1}$  and in the present case the absence of CDD ambiguities follows immediately from our semiclassical results.

Recently an alternative view of the theory has emerged based on the work of Bakas [13] (see also [14]), who showed at the classical level that the CSG Lagrangian with coupling constant  $\lambda^2 = 4\pi/k$  is obtained as a particular gauge fixing of a perturbation (by the first thermal operator) of the  $SU(2)/U(1)$  coset model at level  $k$ , realized as a gauged WZW model. The coset description resolves the apparent pathology of the model signaled by the singularity at  $|\psi| = 1/\lambda$  in the Lagrangian, which is nothing more than a bad choice of coordinates on the gauge-slice, and provides a definition of the path-integral measure. The results of our investigation are completely consistent with this coset description of the model. In particular, the restriction of  $k$  to integer values which emerges in the CSG theory by demanding an S-matrix which is consistent with the semiclassical limit, has a natural explanation: the coset model is only well-defined quantum mechanically if the level is an integer greater than one. Also, finite counter-terms like those described above which are necessary to maintain the factorizability of the S-matrix arise naturally in the coset model as  $1/k$  corrections to the effective action derived from the gauged WZW action. Furthermore, the coset theory at level  $k$  is known to describe  $\mathbf{Z}_k$  parafermions and the corresponding deformation of this conformal field theory by the first thermal operator has been shown to be integrable. It has been conjectured that the resulting off-critical theory is described by the minimal  $a_{k-1}$  S-matrix [12]. In this paper we will focus primarily on the semiclassical quantization of the CSG theory itself and will discuss these connections to the coset model in detail elsewhere [15].

The soliton solutions of the CSG equation carry the charge,  $Q$ , associated with the global  $U(1)$  symmetry of (1). After quantization,  $Q$  is restricted to integer values and the resulting spectrum is a finite tower of massive charged particles. Unlike the SG kink, the CSG soliton does not carry a topological quantum number and there is no topological distinction between the vacuum sector of the theory and the one-soliton sector. A surprising consequence of this is that the elementary particle of the quantum CSG theory can be identified with the  $Q = 1$  soliton itself. In this paper we also calculate the semiclassical S-matrix for the scattering of this particle with a soliton of arbitrary charge. In particular we calculate this scattering amplitude using both representations of the  $Q = 1$  particle of the theory and find agreement, thus confirming the above mentioned identification. As well as the leading-order behaviour of this S-matrix element we are also able to calculate the leading-order positions and residues of its poles. These calculations provide us with several independent semiclassical checks on our conjectured exact S-matrix. The paper is organized as follows: in Section 2 we introduce the CSG model and discuss the soliton solutions of the field equation. We quantize these solutions and obtain the semi-classical spectrum of the model. This section also contains a calculation of the semiclassical phase shift in soliton-soliton scattering. Most of the results in Section 2 are not new and were first obtained in the paper by de Vega and Maillet [9], but we emphasize several novel features of the model which have not been discussed before. Section 3 is devoted to a semiclassical calculation of the meson-soliton S-matrix. In Section 4 we introduce our conjecture for the exact S-matrix of the whole theory and show that the conjectured form agrees with all the known semiclassical results for the theory.

obtained in Sections 2 and 3. Finally we present our conclusions and give a brief account of the consistency of our results with the definition of the theory as a deformed coset model mentioned above.

## 2 Complex sine-Gordon solitons

The classical equation of motion which follows from (1) is

$$\psi + \lambda^2 \psi^* \frac{(\partial_\mu \psi)^2}{1 - \lambda^2 |\psi|^2} + m^2 \psi (1 - \lambda^2 |\psi|^2) = 0. \quad (2)$$

This equation has a two parameter family of time-independent soliton solutions:

$$\psi_S(x; \theta, X) = \frac{e^{i\theta}}{\lambda} \frac{1}{\cosh(m(x - X))}. \quad (3)$$

The symmetry coordinates  $X$  and  $\theta$  arise because the soliton solution breaks the translational invariance and the global U(1) invariance of the Lagrangian (1). They correspond to the centre of mass of the soliton and its orientation in the target space. In addition, equation (2) has explicit time-dependent solutions, labelled by a real parameter  $\alpha$ , which correspond to a soliton rotating in the internal U(1) space with a constant angular velocity  $\omega = m \sin \alpha$ ; setting  $\theta = X = 0$  we have

$$\psi_S(x, t) = \frac{\cos(\alpha)}{\lambda} \frac{\exp(im \sin(\alpha)t)}{\cosh(m \cos(\alpha)x)}. \quad (4)$$

The static solution (3) is recovered by setting  $\alpha = 0$  in the above formula. The classical mass  $M$  and U(1) charge,  $Q$ , are given by

$$M = \int dx \frac{|\dot{\psi}|^2 + |\psi'|^2}{1 - \lambda^2 |\psi|^2} + m^2 |\psi|^2, \quad Q = i \int dx \frac{\psi^* \dot{\psi} - \psi \dot{\psi}^*}{1 - \lambda^2 |\psi|^2}. \quad (5)$$

Evaluated on the time-dependent soliton solutions (4) this yields

$$M = \frac{4m}{\lambda^2} \cos(\alpha), \quad Q = \frac{4}{\lambda^2} \left( \text{sign}[\alpha] \frac{\pi}{2} - \alpha \right). \quad (6)$$

It is instructive to note several unusual features of these solutions. Firstly we note that both the mass and the charge of the soliton *decrease* with the angular velocity of rotation  $\omega$ :

$$M(\omega) = M(0) \sqrt{1 - \frac{\omega^2}{m^2}}, \quad Q = \frac{4}{\lambda^2} \cos^{-1}(\omega), \quad (7)$$

where  $M(0) = 4m/\lambda^2$  is the mass of the static soliton solution (3). This is in sharp contrast with the more familiar behaviour of the Skyrmion, for which

$$M(\omega) = M(0) + \frac{1}{2} \Lambda \omega^2, \quad Q = \Lambda \omega, \quad (8)$$

where  $\Lambda$  is the moment of inertia of the Skyrmion which is independent of the angular velocity at weak coupling. In addition the moment of inertia of the Skyrmion becomes large in the semiclassical limit and correspondingly the rotational contribution to the soliton mass, which goes like  $Q^2/2\Lambda$ , is small in this limit and always positive. In the CSG case, the rotational contribution is of the same order as the static contribution and is negative. In the extreme case where the angular velocity reaches its maximum at  $\alpha = \pi/2$  the rotational contribution exactly cancels the static contribution and the mass (as well as the soliton configuration itself) vanishes.

The second unusual feature is the singular nature of the time-independent solution obtained by setting  $\alpha = 0$ . The static solution (3) is the only solution which attains the maximum value,  $1/\lambda$ ; in this case at  $x = X$ . At this value the factor  $1/(1 - \lambda^2|\psi|^2)$ , which appears in the expressions (5) for the mass and charge of the soliton, becomes singular. Although the mass of this field configuration remains finite and equals  $4m/\lambda^2$ , its  $U(1)$  charge is ill-defined. If we calculate the charge  $Q$  as function of  $\alpha$  by substituting (4) for  $\psi$  in (5), we find that this function exhibits a branch cut which runs through the origin of the complex  $\alpha$ -plane. For real  $\alpha$  the result is given in equation (6) and we see that there is no unique limit as  $\alpha \rightarrow 0$ :

$$\lim_{\alpha \rightarrow 0^\pm} Q(\alpha) = \pm \frac{2\pi}{\lambda^2}. \quad (9)$$

This ambiguity suggests that the  $U(1)$  charge can only be defined in CSG theory modulo  $4\pi/\lambda^2$ . This feature will play an important role in our proposed solution of the model.

As for any Lorentz invariant field equation having soliton solutions, for each of solutions given above we can generate a further one-parameter family of uniformly moving solutions by applying a Lorentz boost with rapidity  $\beta$ . Hence the complete set of one soliton solutions of (2) is

$$\psi_S(x, t; \alpha, \beta, \theta, X) = \frac{\cos(\alpha)}{\lambda} \frac{\exp[i\theta + im \sin(\alpha)(\cosh(\beta)x - \sinh(\beta)t)]}{\cosh[m \cos(\alpha)(\cosh(\beta)(x - X) - \sinh(\beta)t)]}. \quad (10)$$

Having reviewed the classical properties of CSG solitons we will now consider their quantization. As we have explicit time-dependent soliton solutions we can obtain the semiclassical spectrum of soliton states by applying the Bohr-Sommerfeld quantization rule;  $S[\psi_S] + M\tau = 2\pi n$ , where  $n$  is a positive integer,  $\tau = 2\pi/\omega$  is the period of the soliton solution (4) and  $S$  is its action; whence,

$$S[\psi_S] + M\tau = \int_0^\tau dt \int_{-\infty}^{+\infty} dx \frac{2|\psi_S|^2}{1 - \lambda^2|\psi_S|^2} = 2\pi|Q|, \quad (11)$$

and therefore the charge  $Q$  is restricted to integer values. Hence the semiclassical mass spectrum of the charged soliton states is given by,

$$M(Q) = \frac{4m}{\lambda^2} \left| \sin \left( \frac{\lambda^2 Q}{4} \right) \right|, \quad (12)$$

where  $Q = \pm 1, \pm 2, \dots, \pm Q_{\max} = [2\pi/\lambda^2]$  and  $[x]$  denotes the greatest integer less than or equal to  $x$ . The above spectrum is a periodic function of  $Q$  with period  $4\pi/\lambda^2$  which again suggests

that it is correct to think of  $Q$  as an angular variable. The spectrum is similar in form to the spectrum of breather states in the ordinary sine-Gordon theory except here the states are labelled by a conserved U(1) charge which takes negative as well as positive values. The lightest states in the spectrum have charge  $Q = \pm 1$  and mass  $M(1) = m + O(\lambda^4)$ . These quantum numbers are identical to those of the elementary meson of the theory (together with its anti-particle) which is interpolated by the complex scalar field  $\psi$  and we therefore assume that the  $Q = \pm 1$  state is the elementary meson. The state of charge  $Q$  then has an obvious interpretation as a bound-state of  $Q$  mesons. A similar phenomenon occurs for the breathers of the SG theory, where the lightest breather state is identified with the elementary particle of that theory. Thus, in that case, the elementary particle is identified as a bound state of two sine-Gordon kinks. In the CSG case, we have just found that the elementary particle is to be identified as a rotational excitation of the soliton itself. This is possible because the CSG soliton does not carry a topological charge and correspondingly there is no topological distinction between the vacuum sector and the one-soliton sector.

In general each state of charge  $Q$  has a partner of charge  $-Q$  which is identified as its anti-particle. However, an important special case occurs when the coupling constant takes one of the infinite discrete set of special values  $\lambda^2 = 2\pi/N$  where  $N = 1, 2, \dots$ . At this point the number of stable particles in the spectrum changes discontinuously from  $2N-2$  to  $2N$ . The Bohr-Sommerfeld quantization rule means that each state in the quantum theory is associated with a single time-dependent solution of the CSG field equation which has the same mass and charge. However, precisely at these special values the two soliton states with charge  $Q = \pm Q_{\max} = \pm N$  are associated with the *same* solution; the static soliton (3). As discussed above the U(1) charge of this classical solution is double-valued; having an ambiguity of exactly  $2N$ . We propose a resolution for this puzzle; a single valued spectrum is obtained by identifying U(1) charges modulo  $2N$  so that the states  $\pm N$  actually correspond to the same particle. More generally for the values  $\lambda^2 = 4\pi/k$  where  $k$  is an integer  $> 1$ , the U(1) charge is identified modulo  $k$  and therefore should more properly be thought of as a  $\mathbf{Z}_k$  charge. In the following we will show that this assumption leads to a completely consistent quantum theory.

One-loop quantum corrections to the semiclassical spectrum (12) were computed by de Vega and Maillet [9]. They found that the only effect of these corrections was an overall finite renormalization of the coupling constant:

$$\lambda^2 \rightarrow \lambda_R^2 = \frac{\lambda^2}{1 - \lambda^2/4\pi}. \quad (13)$$

This finite renormalization of the coupling is familiar from the SG theory where the resulting spectrum turns out to be exact. In [9] it is conjectured that, provided suitable finite counter-terms are added to the action to maintain integrability at the quantum level, the one-loop spectrum for the CSG theory is also exact. In the following, we will assume that this is the case and we will also assume that the S-matrix describing the scattering of the charged solitons described above is exactly factorizable. We will show that these assumptions imply a certain minimal form for the S-matrix of the theory and check that this minimal form passes a series of stringent semiclassical

tests. However, we shall only be able to find a consistent S-matrix for which the bootstrap closes when the renormalized coupling constant takes one of the special values described above;  $\lambda_R^2 = 4\pi/k$ .

The classical CSG equation is completely integrable and explicit solutions which describe the scattering of two solitons of arbitrary charge can be constructed, either by the method of Hirota [7], or by the inverse scattering method [9]. As mentioned above, the solitons undergo completely elastic scattering with no change in shape, the only result of the collision being a time delay. The analytic expressions for these solutions are extremely cumbersome, we refer the reader to [9] for details. However, the time delay for soliton scattering can be extracted easily from the large-time asymptotics of these expressions. Consider two solitons with rapidities  $\beta_1, \beta_2$  and internal parameters  $\alpha_1$  and  $\alpha_2$ . The centre-of-momentum (COM) frame for the two solitons can be chosen by imposing the relation;  $\sinh(\beta_1) \cos(\alpha_1) = -\sinh(\beta_2) \cos(\alpha_2)$ . In the COM frame both solitons experience the same time delay due to the collision:

$$\Delta t(\beta_1, \beta_2, \alpha_1, \alpha_2) = \frac{2}{m \sinh(\beta_1) \cos(\alpha_1)} \log \left| \frac{\sinh(\Delta\beta + i\Delta\alpha)}{\cosh(\Delta\beta + i\bar{\alpha})} \right|, \quad (14)$$

where  $\Delta\beta = (\beta_1 - \beta_2)/2$ ,  $\Delta\alpha = (\alpha_1 - \alpha_2)/2$  and  $\bar{\alpha} = (\alpha_1 + \alpha_2)/2$ . The semi-classical phase shifts  $\delta(E)$  for soliton-soliton scattering at total COM energy  $E$ , are completely determined by the time-delay via the WKB formula:

$$\delta(E) = \frac{1}{2} n_B \pi + \frac{1}{2} \int_{E_{\text{Th}}}^E dE' \Delta t(E'), \quad (15)$$

where  $n_B$  is the number of quantized breather states below the threshold energy,  $E_{\text{Th}}$ . As there are no known breather solutions of the CSG equation we will assume that  $n_B = 0$ <sup>1</sup>. This semi-classical formula for the phase shift was derived in the context of soliton scattering by Jackiw and Woo [16]. The COM energy of the two soliton scattering solution is just given by the sum of the two soliton energies which is conveniently written as

$$E = \frac{4m}{\lambda^2} (\cos(\alpha_1) \cosh(\bar{\beta} + \Delta\beta) + \cos(\alpha_2) \cosh(\bar{\beta} - \Delta\beta)), \quad (16)$$

with  $\bar{\beta} = (\beta_1 + \beta_2)/2$ . Hence the semiclassical phase shift is given by;

$$\delta(\Delta\beta) = \frac{4}{\lambda^2} \int_0^{\Delta\beta} dx \log \left[ \frac{\sinh(x + i\Delta\alpha) \sinh(x - i\Delta\alpha)}{\cosh(x + i\bar{\alpha}) \cosh(x - i\bar{\alpha})} \right]. \quad (17)$$

It is easily checked that this expression agrees exactly with formula (4.8) of ref [9] which was derived by slightly different means. An alternative and, for our purposes, more convenient integral representation for the semiclassical phase shift can be derived by changing the contour of integration in (17):

$$\delta(\Delta\beta) = -\frac{2i}{\lambda^2} \int_{2\Delta\alpha}^{\pi-2\bar{\alpha}} d\eta \log \left[ \frac{e^{i\eta+2\Delta\beta} - 1}{e^{i\eta} - e^{2\Delta\beta}} \right]. \quad (18)$$

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<sup>1</sup>Because the equations of motion (2) depend on  $\psi$  and  $\psi^*$ , breather solutions cannot be obtained by analytic continuation to imaginary values of  $\Delta\beta$  as is the case in ordinary SG theory. On this point we are in disagreement with [7].

### 3 Meson-soliton scattering

The aim of our investigation is to calculate the S-matrix for the charged particles described in the previous section. Since the  $Q = 1$  particles can be represented either as solitons or as elementary mesons means that we can apply two separate methods for calculating the two-body scattering amplitude for these particles with the states of higher charge. Firstly, the result (17) of the previous section and the formula (6) for the soliton charges imply that the semiclassical phase shift for meson soliton scattering is given by

$$\delta(\theta) = -\pi - \frac{2i}{\lambda^2} \int_{\frac{\lambda^2}{4}(Q-1)}^{\frac{\lambda^2}{4}(Q+1)} d\eta \log \left[ \frac{\sinh\left(\frac{\theta}{2} + i\frac{\eta}{2}\right)}{\sinh\left(\frac{\theta}{2} - i\frac{\eta}{2}\right)} \right]. \quad (19)$$

where  $\theta = \beta_1 - \beta_2$ . The usual simplifying feature of the semi-classical limit for meson-soliton scattering is that the soliton becomes very massive. However in the CSG case this is not true for all the states in the semi-classical spectrum (12). For this reason we will restrict our attention to the case where the mass of the charge- $Q$  particle grows like  $\lambda^{-2}$  in the semi-classical limit. This means choosing a value of  $Q$  which remains a finite distance from the top of the tower (12) in this limit; hence we take  $Q \sim \lambda^{-2}$ . The leading semi-classical behaviour of the S-matrix,  $\mathcal{S}(\theta) = \exp(2i\delta)$  is then given by the simple formula

$$\mathcal{S}_{1,Q}(\theta) = \frac{\sinh^2\left(\frac{\theta}{2} + i\frac{\lambda^2 Q}{8}\right)}{\sinh^2\left(\frac{\theta}{2} - i\frac{\lambda^2 Q}{8}\right)} + O(\lambda^2). \quad (20)$$

In fact it is also possible to obtain this formula for  $\mathcal{S}(\theta)$  directly from the mesonic representation of the  $Q = 1$  particle. Let us expand the field around the rest-frame soliton solution (4) as  $\psi = \psi_S(x; \alpha) + \delta\psi$ . The Lagrangian density becomes

$$\mathcal{L}[\psi] = \mathcal{L}[\psi_S] + \delta\psi^* \hat{M} \delta\psi + O(\delta\psi^3), \quad (21)$$

where  $\hat{M} = (\delta^2 \mathcal{L} / \delta\psi \delta\psi^*)[\psi_S(x; \alpha)]$  is a Hermitian differential operator. As is well known, the continuous part of the spectrum of  $\hat{M}$  corresponds to the scattering of a single meson off the background soliton field configuration. The phase shifts of these scattering eigenmodes have been found by de Vega and Maillet using the inverse scattering method. In our notation, their equation (3.18a) for the phase shift of a positively charged meson scattering off a soliton of charge  $Q$  reads

$$\delta_+(\theta) = 2 \tan^{-1} \left[ \frac{\sin\left(\frac{\lambda^2 Q}{4}\right) \sinh(\theta)}{\cos\left(\frac{\lambda^2 Q}{4}\right) \cosh(\theta) - 1} \right]. \quad (22)$$

It is straightforward to check that the resulting S-matrix element,  $\exp(2i\delta_+)$ , coincides exactly with the formula (20) obtained above. This agreement provides further confirmation for the identification of the  $Q = 1$  soliton state with the elementary meson of the theory.



Although the leading order  $S$ -matrix (20) for meson-soliton scattering is unitary and completely elastic by construction it lacks several features we would expect from the exact  $S$ -matrix of the theory. In particular, given the existence of stable particles of charge  $Q + 1$  and  $Q - 1$  in the semiclassical spectrum (12) for  $0 < Q < Q_{\max}$ , we would expect these particles to arise as intermediate states in the scattering amplitude in the direct and crossed channel respectively. Assuming the one-loop semiclassical spectrum is exact, the kinematics of the processes illustrated in Figs 1 a) and b) dictates the presence of simple poles in the scattering amplitude  $\mathcal{S}_{1,Q}(\theta)$  at  $\theta = i\lambda_R^2(Q \pm 1)/4$  (recall that  $\lambda_R^2 = \lambda^2 + O(\lambda^4)$ ). In fact, rather than these simple poles, the leading semi-classical expression has a double pole at  $\theta = i\lambda^2 Q/4$ . The reason for this is simply that for a soliton state a fixed distance from the top of the tower with charge  $Q \sim \lambda^{-2}$  the splitting between the direct and crossed channel poles is a sub-leading effect; at leading order these poles merge to form a double pole.

If we assume that the exact  $S$ -matrix for meson-soliton scattering is factorizable then its form is highly constrained (for a review of elastic  $S$ -matrices in two dimensions see [10]). In particular, the absence of particle creation and of reflection for particles of different masses means that the unitarity condition takes the simple form:  $\mathcal{S}(\theta)\mathcal{S}(-\theta) = 1$ . This condition together with the assumption of analyticity implies that the  $S$ -matrix must be a product of factors of the form

$$F_x(\theta) = \frac{\sinh\left(\frac{\theta}{2} + i\frac{\lambda_R^2 x}{8}\right)}{\sinh\left(\frac{\theta}{2} - i\frac{\lambda_R^2 x}{8}\right)}, \quad (23)$$

where we have chosen a convenient normalization for the label  $x$ . Each factor has a simple pole at  $\theta = ix\lambda_R^2/4$ . Clearly the minimal choice for the  $S$ -matrix which has the two required simple poles is

$$\mathcal{S}_{1,Q}(\theta) = F_{Q-1}(\theta)F_{Q+1}(\theta). \quad (24)$$

It is easy to check that this expression agrees with our semiclassical result (20) at leading order in  $\lambda^2$ . In the following we will provide further evidence that (24) is the exact  $S$ -matrix for meson soliton scattering in CSG theory.

As discussed above the simple poles in the meson-soliton scattering amplitude correspond to the process where the soliton undergoes a transition from its excited state of charge  $Q$  to the state of charge  $Q \pm 1$  by absorbing or emitting a single meson. These processes are completely analogous to the contribution of the  $\Delta$  resonance to pion-nucleon scattering in the Skyrme model [2, 17]. Of course, in the present case, the mass splitting between adjacent soliton states in the tower is always less than the meson mass, so these processes are below threshold and can only be realized at imaginary values of the external momenta. In the Skyrme model, although the mass splitting between the  $\Delta$  and the nucleon is a subleading effect in the semi-classical limit, the pion mass is an independent parameter of the theory, and hence the  $\Delta$  resonance can be kept above threshold. Because the splitting is subleading, a straightforward semi-classical analysis of the pion-Skyrmion  $S$ -matrix by finding the continuous spectrum of the small fluctuation operator, exactly analogous to the derivation of the result (22), does not find the  $\Delta$  contribution. In recent

work, one of the authors [17] has developed a method which allows a systematic semi-classical evaluation of the width of the  $\Delta$  resonance. The method is equally applicable to processes below threshold and can be used to calculate the positions and residues of the S-matrix poles corresponding to the transitions  $Q \rightarrow Q \pm 1$  providing a further check on the proposed exact S-matrix. The first step is to calculate the leading semiclassical contribution to the two-point Green's function of the meson field in the one-soliton sector. This can be accomplished by applying the saddle-point approximation to an appropriate Feynman path integral; some details of this calculation are given in the Appendix. The bound-state contribution to the T-matrix can then be obtained by applying the LSZ reduction formula to this Green's function, the result is

$$\mathcal{T}_{1,Q}(\omega) = -\frac{16m^2}{\lambda^2} \left[ \frac{\sin^4\left(\frac{\lambda^2(Q+\frac{1}{2})}{4}\right)}{\omega - M(Q+1) + M(Q)} - \frac{\sin^4\left(\frac{\lambda^2(Q-\frac{1}{2})}{4}\right)}{\omega - M(Q) + M(Q-1)} \right], \quad (25)$$

where  $\omega$  is the energy of the incoming meson in the rest frame of the soliton. The two terms in this expression correspond directly to the processes shown in Figures 1 a) and 1 b) respectively. With the correct normalization [19] the corresponding contribution to the S-matrix is given by

$$\mathcal{S}_{1,Q} = 1 + \frac{\mathcal{T}_{1,Q}(\omega)}{2ik}, \quad (26)$$

where  $k$  is the meson momentum. We now compare this result with our conjecture for the exact meson soliton S-matrix (24). As we are working in the rest frame of the soliton, the rapidity difference  $\theta$  is simply the rapidity of the incoming meson; thus we set  $\omega = m \cosh(\theta)$  and  $k = m \sinh(\theta)$ . From (25) we find that the S-matrix has poles at  $\theta = i\lambda^2(Q \pm \frac{1}{2})/4$  with residues

$$\mathcal{R}_{\pm} = \pm \frac{8i}{\lambda^2} \sin^2\left(\frac{\lambda^2(Q \pm \frac{1}{2})}{4}\right). \quad (27)$$

As mentioned above the correct positions of the poles are dictated by relativistic kinematics to be at  $\theta = i\lambda_R^2(Q \pm 1)/4$ . The discrepancy with the saddle-point result is due to our neglect of the translational motion of the soliton and the renormalization of the coupling, both of which are subleading effects as long as  $Q \sim \lambda^{-2}$ . The proposed exact result has a poles at the above mentioned positions with residues  $\mathcal{R}_{\pm}^{exact} = \pm(8i/\lambda_R^2) \sin^2(\lambda_R^2 2Q/4)$ . Clearly the pole positions and residues agree to leading order in the semi-classical limit  $\lambda^2 \rightarrow 0$ .

## 4 Soliton-soliton scattering

In the previous section we found evidence to support a completely factorizable form (24) for the scattering amplitude of a soliton of unit charge with a soliton of charge  $Q \sim \lambda^{-2}$ . This encourages us to extend this conjecture to the scattering amplitude for two solitons of arbitrary charges. As discussed in the previous section, unitarity and analyticity imply that the factorizable S-matrix element must be a product of factors  $F_{x(i)}(\theta)$  defined in (23), for some set of real numbers  $\{x(i)\}$ . As each of these factors supplies a simple pole, the minimal choice for the S-matrix is determined by the set of singularities which are consistent with the spectrum (12) (with the one-loop replacement  $\lambda^2 \rightarrow \lambda_R^2$ ).

Let us first consider the case of two solitons with charges  $Q_1 \geq Q_2 > 0$  such that  $Q_1 + Q_2 \leq Q_{\max}$ . In this case the semi-classical spectrum tells us that there are stable particles with charges  $Q_1 + Q_2$  and  $Q_1 - Q_2$  and we expect these states to arise as intermediate states in the direct and crossed channels respectively (see Figs 2 a) and b)). It follows that there should be simple poles in the scattering amplitude at  $\theta = i\lambda_R^2(Q_1 \pm Q_2)/4$  and thus we include the factors  $F_{Q_1+Q_2}(\theta)$  and  $F_{Q_1-Q_2}(\theta)$ . However, simple poles corresponding to bound states are not the only kinematically allowed singularities of the S-matrix. In four dimensions, anomalous thresholds due to two-particle intermediate states lead to cuts in the the amplitude, in two dimensions these singularities become double poles. The rules for determining the positions of these double poles were discussed in detail by Coleman and Thun [20]. In particular, these authors used their rules to explain the positions of double poles in the breather S-matrix of the SG model. In the CSG case the kinematics of the processes illustrated in Fig 4, in which the intermediate state consists of two solitons with charges  $Q_2 - n$  and  $Q_1 + n$  for  $n = 1, 2, \dots, Q_2 - 1$ , leads to a total of  $Q_2 - 2$  double poles in the amplitude at  $\theta = i\lambda_R^2(Q_1 - Q_2 + 2n)/4$ . The minimal choice for a factorizable S-matrix which has all the required poles is therefore given by

$$\mathcal{S}_{Q_1, Q_2}(\theta) = F_{Q_1-Q_2}(\theta) \left[ \prod_{n=1}^{Q_2-1} F_{Q_1-Q_2+2n}(\theta) \right]^2 F_{Q_1+Q_2}(\theta). \quad (28)$$

The particles in the semi-classical spectrum occur in pairs of opposite charge and it is natural to interpret the state with charge  $-Q$  as the anti-particle of the state with charge  $Q$ . In this case the extension of our proposed S-matrix to negative values of  $Q_1$  and  $Q_2$  is completely determined by crossing symmetry:

$$\begin{aligned} \mathcal{S}_{Q_1, -Q_2}(\theta) &= \mathcal{S}_{Q_1, Q_2}(i\pi - \theta), \\ \mathcal{S}_{-Q_1, -Q_2}(\theta) &= \mathcal{S}_{Q_1, Q_2}(\theta). \end{aligned} \quad (29)$$

We will now check that the above S-matrix reduces to the semi-classical expression derived in Section 1 above. Unlike the previous section, we will consider two solitons of fixed U(1) charge in the semiclassical limit:  $Q_1, Q_2 \sim \lambda^0$ . We first consider the case  $Q_1 \geq Q_2 > 0$ . Using the integral representation (18) for the semiclassical phase shift we find

$$\mathcal{S}_{Q_1, Q_2}(\theta) = \exp \left( \frac{4}{\lambda^2} \int_{\frac{\lambda^2}{4}(Q_1-Q_2)}^{\frac{\lambda^2}{4}(Q_1+Q_2)} d\eta \log \left[ \frac{\sinh \left( \frac{\theta}{2} + i\frac{\eta}{2} \right)}{\sinh \left( \frac{\theta}{2} - i\frac{\eta}{2} \right)} \right] \right) + O(\lambda^4). \quad (30)$$

Defining the discrete set of points  $\eta_j = \lambda^2(Q_1 - Q_2)/4 + j\lambda^2/2$ , for integers  $j$  with  $0 \leq j \leq Q_2$ , the definition of the Riemann integral of an arbitrary function  $f(\eta)$  is

$$\int_{\frac{\lambda^2}{4}(Q_1-Q_2)}^{\frac{\lambda^2}{4}(Q_1+Q_2)} d\eta f(\eta) = \frac{\lambda^2}{4} \sum_{j=1}^{Q_2} [f(\eta_j) + f(\eta_{j-1})] + O(\lambda^6). \quad (31)$$

The equality of our proposed S-matrix  $\mathcal{S}$  and the semiclassical result (30) to  $O(\lambda^4)$  follows immediately from setting  $f(\eta) = \log(F_{\frac{4\eta}{\lambda^2}}(\theta))$  in the above relation. This agreement can immediately

be extended to the cases  $Q_1 < 0$  and/or  $Q_2 < 0$  by noting that the semiclassical phase shift (18) satisfies the crossing relationship,

$$\delta_{Q_1, -Q_2}(\theta) = \delta_{Q_1, Q_2}(i\pi - \theta) \quad (32)$$

To obtain the complete S-matrix of the theory it is necessary to relax our requirement that  $Q_1 + Q_2 \leq Q_{\max}$ . In fact the agreement between our conjectured S-matrix and the semiclassical expression (30) extends to all values of  $Q_1$  and  $Q_2$  and, in this sense, there is no obstacle to applying the formulae (28) everywhere. However, there is now a problem in the sense that the S-matrix elements  $\mathcal{S}_{Q_1, Q_2}(\theta)$  with  $Q_1 + Q_2 > Q_{\max}$  have a simple pole at  $\theta = i\lambda_R^2(Q_1 + Q_2)/4$  which cannot be explained by the exchange in either the direct or crossed channels of one the solitons in the spectrum. Either the spectrum must be enlarged to include new states or these simple poles must be accounted for via another mechanism. We will defer a discussion of this problem till the last section.

Fortunately, in the special cases discussed in Section 1, where  $\lambda_R^2 = 4\pi/k$  for some integer  $k > 1$ , we will see that all the poles in the S-matrix (28), for arbitrary values of  $Q_1$  and  $Q_2$ , *can* be explained in terms of the particles in the semiclassical spectrum alone. This explanation implies that the U(1) charge is only conserved modulo  $k$  which is directly related to the ambiguity (9). In this case the spectrum consists of  $k - 1$  particles with  $Q = \pm 1, \pm 2, \dots, \pm(k - 1)/2$  for  $k$  odd and  $Q = \pm 1, \pm 2, \dots, \pm(k - 2)/2, k/2$  for  $k$  even. (In the later case recall that the states with charge  $\pm k/2$  are identified.) The masses of these particles are given by

$$M(Q) = \frac{km}{\pi} \left| \sin \left( \frac{Q\pi}{k} \right) \right|. \quad (33)$$

This is the spectrum of the minimal S-matrix associated with the Lie algebra  $a_{k-1}$  [22] and indeed, for  $\lambda^2 = 4\pi/k$ , our conjecture (28) is precisely that S-matrix. In this case, for any  $Q_1$  and  $Q_2$ , there is always two particles in the spectrum (33) with charge  $Q_3 = Q_1 \pm Q_2 \bmod k$  which correspond to the simple poles in  $\mathcal{S}_{Q_1, Q_2}(\theta)$ . In fact, it is well known that the minimal  $a_{k-1}$  S-matrix defines a completely consistent theory in which all poles, both simple and double, can be explained in terms of the above spectrum (see for example [21]).

## 5 Conclusions

The main result of this paper is as follows: if one assumes that the CSG Lagrangian with  $\lambda_R^2 = 4\pi/k$  defines a completely integrable quantum field theory for every integer  $k > 1$ , then the minimal  $a_{k-1}$  S-matrix is the minimal S-matrix which is consistent with the semiclassical spectrum of the model. We therefore conjecture that this S-matrix together with the one-loop semiclassical spectrum constitute the exact solution of the CSG model for these values of the coupling. Our conjectured S-matrix successfully reproduces the leading semiclassical behaviour of the scattering of two solitons of arbitrary charges. In addition the conjectured form agrees both with the leading semiclassical expression for the meson-soliton scattering amplitude and also with the leading order calculation of the positions and residues of the poles in this amplitude.

These semiclassical checks clearly imply that any non-minimal factors in the exact S-matrix must vanish at  $O(\lambda^2)$ . It is then straightforward to argue that any CDD factors which are consistent with the  $a_{k-1}$  bootstrap equations cannot be present. For instance, for the scattering of two particles of charge  $Q = 1$ , the CDD factor which is consistent with the bootstrap equations for  $a_{k-1}$  has the form

$$\frac{\sinh\left(\frac{\theta}{2} - \frac{i\pi}{2k}(2 - \alpha)\right) \sinh\left(\frac{\theta}{2} - \frac{i\pi}{2k}\alpha\right)}{\sinh\left(\frac{\theta}{2} + \frac{i\pi}{2k}(2 - \alpha)\right) \sinh\left(\frac{\theta}{2} + \frac{i\pi}{2k}\alpha\right)}, \quad (34)$$

where  $\alpha(\lambda^2)$  is some function of the coupling constant such that  $0 < \alpha < 2$ , in order that no additional poles are introduced onto the physical strip. Notice that such a CDD factor always has a contribution at  $O(1/k)$ , i.e.  $O(\lambda^2)$ , whatever the functional dependence of  $\alpha$ , and is therefore ruled out by our tree-level results.

A final possible semiclassical check on our proposed S-matrix is to calculate the scattering amplitude for two mesons using ordinary perturbation theory and compare it with our conjectured form  $\mathcal{S}_{1,1}(\theta) = F_2(\theta)$ . This scattering amplitude has been calculated to one loop by de Vega and Maillet [8]. Our conjecture agrees with their calculation at  $O(\lambda^2)$  but differs at  $O(\lambda^4)$ . However, the one-loop contribution of Ref [8] depends crucially on the form of the finite counter-terms added to the action and we believe that these terms should be re-evaluated in the light of the definition of the model as a perturbed coset theory.

For other values of the coupling the S-matrix elements have simple poles which would seem to require at least the existence of additional states in order to be explained. These additional states are not realized semiclassically; for instance, we have argued that there are no breather solutions. Another possibility is that the scattering of a soliton  $Q$  with  $-Q$  has a reflection amplitude away from the discrete values of the coupling. This idea is motivated by analogy with the SG theory where the soliton S-matrix in general is non-diagonal, except at the threshold values of the coupling where the total S-matrix for the solitons and breathers becomes the minimal  $d_k$  S-matrix, up to signs [21]. However, after pursuing these two possibilities, we have not found it possible to define a consistent factorizable S-matrix for these other values of the coupling constant. As we mentioned in the introduction this has a perfectly natural explanation in terms of the realization of the model as a perturbed coset model which is only well-defined quantum mechanically if the level  $k$  is an integer  $> 1$ . The perturbation of the  $SU(2)/U(1)$  coset model at level  $k$  by the first thermal operator having dimensions  $\Delta = \bar{\Delta} = 2/(k+2)$  has been discussed in [12]. Following the methods of Zamolodchikov [24] the perturbation is shown to be integrable and the S-matrix has been conjectured to be the  $a_{k-1}$  minimal S-matrix. This is precisely the result of our calculation. Notice that the hidden  $\mathbf{Z}_k$  structure of the solitons manifests the  $\mathbf{Z}_k$  structure of the parafermions, a fact which deserves to be studied in more detail. Finally we note that the part of the finite renormalization of the coupling (13) which comes from purely from the Lagrangian (1)<sup>2</sup> corresponds exactly to an integer shift in the level  $k \rightarrow k+2$  which is encountered in gauged WZW models (see [25] and references therein). A detailed discussion of this correspondence will be presented elsewhere [15].

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<sup>2</sup>In other words, omitting the contribution of the finite one-loop counter-term [9].

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## Appendix

The purpose of this Appendix is to calculate the positions and residues of the simple poles in the meson-soliton S-matrix to leading order in the semiclassical approximation. The calculation is essentially a direct application of the method developed in [17] and we refer the reader to this paper for further details. The basic object in quantum field theory from which the meson-soliton scattering amplitude can be extracted is a two-point Green's function for the meson field  $\psi$  evaluated not in the vacuum sector but rather in the one-soliton state of charge  $Q$ :

$$G_Q(x, x'; t - t') = \langle Q | \psi(x, t) \psi^*(x', t') | Q \rangle. \quad (35)$$

Following the collective coordinate approach of Gervais, Jevicki and Sakita [23], we write this Green's function as a phase-space path integral over  $\psi$  and its conjugate momentum;  $\pi = \delta\mathcal{L}/\delta\dot{\psi}^*$ .

$$G_Q = \int \mathcal{D}\psi \mathcal{D}\psi^* \mathcal{D}\pi \mathcal{D}\pi^* \Phi_Q^*[\psi(x, T)] \Phi_Q[\psi(x, -T)] \psi(x, t) \psi^*(x', t') \exp \left( i \int d^2x \pi \dot{\psi}^* + \pi^* \dot{\psi} - \mathcal{H} \right), \quad (36)$$

where  $\Phi_Q$  is the wave-functional for the soliton state of charge  $Q$  which will be identified explicitly below. The Hamiltonian density is given by  $\mathcal{H} = \pi^* g^{-1}(|\psi|) \pi + V(|\psi|, |\psi'|)$  where

$$g(|\psi|) = \frac{1}{1 - \lambda^2 |\psi|^2} \quad V(|\psi|, |\psi'|) = g(|\psi|) |\psi'|^2 + m^2 |\psi|^2. \quad (37)$$

The CSG Lagrangian differs from the model considered in [17] through the presence of the factor  $g(|\psi|)$  which appears in the kinetic term. The generalization of the analysis to include exactly such a factor, which can be thought of as a target-space metric, was given in Appendix A of [18]. The essential result of reference [17] is that the leading contribution of the intermediate states  $|Q \pm 1\rangle$  to meson-soliton scattering can be extracted by evaluating this path-integral in the semi-classical saddle-point approximation. The saddle-point equation is just the classical field equation itself and so the relevant saddle-point field configurations  $\psi_{sp}$  are the time-dependent one-soliton solutions (4) which are parametrized by a collective coordinate  $\theta$  and its conjugate momentum which is the  $U(1)$  charge,  $Q$ :

$$\psi_{sp}(x, t; \theta, Q) = \frac{e^{i\theta}}{\lambda^2} \frac{\sin\left(\frac{\lambda^2 Q}{4}\right)}{\cosh\left(m \sin\left(\frac{\lambda^2 Q}{4}\right) x\right)}. \quad (38)$$

As in reference [17], we have neglected the translational motion of the soliton which gives sub-leading recoil corrections to the S-matrix elements. At leading order the wave-functional for

the state  $|Q\rangle$  is a function of the collective coordinate  $\theta$  only; up to an irrelevant normalization  $\Phi_Q = \exp(iQ\theta)$  and the saddle-point contribution to the path integral is just

$$G_Q = \int \mathcal{D}\theta(t) \mathcal{D}Q(t) \exp[iQ(\theta(-T) - \theta(T))] \psi_{sp}(x, t; \theta(t), Q(t)) \psi_{sp}^*(x', t'; \theta(t'), Q(t')) \\ \times \exp\left(\int_{-T}^T dt Q \dot{\theta} - \frac{4m}{\lambda^2} \sin\left(\frac{\lambda^2 Q}{4}\right)\right). \quad (39)$$

Now the problem has been reduced to one-dimensional quantum mechanics of the Hamiltonian  $\hat{H} = (4m/\lambda^2) \sin(\lambda^2 \hat{Q}/4)$  with the canonical commutation relation  $[\hat{\theta}, \hat{Q}] = i$ . Thus,

$$G_Q(x, x'; t - t') = \langle Q | \psi_{sp}(x, t; \hat{\theta}, \hat{Q}) \psi_{sp}^*(x', t'; \hat{\theta}, \hat{Q}) | Q \rangle, \quad (40)$$

where the operator ordering ambiguity inherent in the saddle-point field operator  $\hat{\psi}_{sp}(x, t) = \psi_{sp}(x, t; \hat{\theta}, \hat{Q})$  is resolved by choosing the Weyl ordering prescription [17]. It is easy to see that this operator, being proportional to  $\exp(i\hat{\theta})$ , only gives a non-zero contribution when evaluated between states whose charge differs by one unit. Resolving the time-ordering and inserting a complete set of states we find

$$G_Q(x, x'; t - t') = \Theta(t' - t) \langle Q | \hat{\psi}_{sp}(x, t) | Q + 1 \rangle \langle Q + 1 | \hat{\psi}_{sp}^*(x', t') | Q \rangle \\ + \Theta(t - t') \langle Q | \hat{\psi}_{sp}^*(x', t') | Q - 1 \rangle \langle Q - 1 | \hat{\psi}_{sp}(x, t) | Q \rangle. \quad (41)$$

The two terms in this expression correspond directly to the forward and crossed channel processes shown in Figures 1 a) and 1 b) respectively. The expectation values in (41) can be evaluated using elementary quantum mechanics; from (38) we find

$$\langle Q_1 | \hat{\psi}_{sp}(x, t) | Q_2 \rangle = \delta_{Q_1+1, Q_2} \frac{1}{\lambda} \frac{\sin\left(\frac{\lambda^2 \bar{Q}}{4}\right)}{\cosh\left(m \sin\left(\frac{\lambda^2 \bar{Q}}{4}\right) x\right)} \exp[it(M(Q_2) - M(Q_1))], \quad (42)$$

where  $\bar{Q} = (Q_1 + Q_2)/2$  and  $M(Q) = (4m/\lambda^2) \sin(\lambda^2 Q/4)$ . The midpoint combination  $\bar{Q}$  arises because of the Weyl ordering of the field operator. All that remains is to extract the contribution to the T-matrix via the LSZ reduction formula:

$$\mathcal{T} = \int \frac{dx dt}{\sqrt{2\omega}} \exp(-ikx + i\omega t) (\partial_t^2 - \partial_x^2 + m^2) \\ \times \int \frac{dx' dt'}{\sqrt{2\omega'}} \exp(ik'x' - i\omega't') (\partial_{t'}^2 - \partial_{x'}^2 + m^2) G_Q(x, x'; t - t'), \quad (43)$$

where  $\omega^2 = k^2 + m^2$  and  $\omega'^2 = k'^2 + m^2$ . The most important point of the analysis given in [17] was to show that  $G_Q$  has poles in momentum space exactly on the mass shells of the incoming and outgoing mesons and thus yields a non-vanishing contribution to  $\mathcal{T}$ . To see this note that the saddle point field  $\psi_{sp}$  has the asymptotic behaviour

$$|\psi_{sp}(x, t)| \rightarrow \frac{2}{\lambda} \sin\left(\frac{\lambda^2 Q}{4}\right) \exp\left(-m \sin\left(\frac{\lambda^2 Q}{4}\right) |x|\right), \quad (44)$$

as  $|x| \rightarrow \infty$ . This means that we can write the expectation value (42) as

$$\langle Q | \hat{\psi}_{sp}(x, t) | Q + 1 \rangle = \int \frac{d\omega dk}{(2\pi)^2} \frac{\mathcal{A}(k)}{k^2 + m^2 \sin^2\left(\frac{\lambda^2 \bar{Q}}{4}\right)} \delta(\omega - M(Q + 1) + M(Q)), \quad (45)$$

where the residue  $\mathcal{A}(k)$  remains finite at  $k^2 = -m^2 \sin^2(\lambda^2 \bar{Q}/4)$ . Hence the Fourier transform of the meson Greens function has a pole at this point. The meson mass shell condition is  $k^2 = m^2 - \Delta M^2$  where  $\Delta M = M(Q + 1) - M(Q)$ . It is straightforward to check that this value of  $k^2$  coincides with the position of the pole up to  $\lambda^4$ . Hence the momentum-space Green's function  $\tilde{G}_Q$  has a pole on the meson mass-shell and yields a non-vanishing T-matrix element. The result is

$$\mathcal{T}_{1,Q}(\omega) = -\frac{16m^2}{\lambda^2} \left[ \frac{\sin^4\left(\frac{\lambda^2(Q+\frac{1}{2})}{4}\right)}{\omega - M(Q + 1) + M(Q)} - \frac{\sin^4\left(\frac{\lambda^2(Q-\frac{1}{2})}{4}\right)}{\omega - M(Q) + M(Q - 1)} \right]. \quad (46)$$

As we show in Section 3, the resulting contribution to the S-matrix has two simple poles whose positions and residues match those of our conjectured scattering amplitude at leading order. Due to the cancellation of the leading order in  $\lambda$  between the two terms in the above expression, this contribution to the scattering amplitude is of  $O(\lambda^0)$ . In fact, there is also another contribution to the S-matrix at this order in the semiclassical approximation which comes from the meson propagator in the soliton background [18]. By analogy with a similar analysis for the  $\phi^4$  kink [19] we expect that this term, when added to the  $O(\lambda^0)$  part of (25), reproduces the naive leading-order result given in the text, equation (20).

## Figure Captions

Figure 1: Contributions of the states with charges  $Q \pm 1$  to the meson-soliton scattering amplitude.

Figure 2: Contributions of the states with charges  $Q_1 \pm Q_2$  to the scattering amplitude for two solitons with charges  $Q_1$  and  $Q_2$ .

Figure 3: Contribution of two-particle intermediate states which produce anomalous thresholds in the soliton-soliton scattering amplitude.

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